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# Second-order boundary layers for steady, incompressible, three-dimensional stagnation point flows

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## INTRODUCTION

It is well known that the classical boundary-layer theory represents the asymptotic solution of the Navier–Stokes equations for large Reynolds numbers. The succeeding approximation is the second-order boundary layer one which includes the effects of surface curvature, vorticity interaction and boundary-layer displacement. The second-order boundary-layer effects become important when the boundary-layer thickness becomes comparable with the characteristic body length. Van Dyke [1] and Gersten and Gross [2] have given an excellent survey of higher-order boundary layers.

The second-order boundary-layer effects on the steady, laminar, incompressible, three-dimensional stagnation point flow with or without mass transfer was considered by Papenfuss [3, 4] for nodal point flows where only the curvature and displacement effects were taken into account. Subsequently, Gersten *et al.* [5] extended the foregoing analysis to include the effect of Prandtl number without mass transfer in nodal point region taking into account only curvature effect. It may be remarked that all the second-order boundary-layer effects in the saddle point region and vorticity interaction effect in the nodal point region have not been considered so far.

The aim of this study is to consider the combined effect of Prandtl number and mass transfer on the second-order boundary layers in both nodal and saddle point regions of a three-dimensional body in the neighbourhood of the stagnation point. The governing equations have been solved using an implicit finite-difference scheme. The results have been compared with those available in the literature.

## GOVERNING EQUATIONS

The steady, laminar, incompressible boundary-layer flow with mass transfer in the stagnation region of a three-dimensional body having two planes of symmetry is considered (Fig. 1). The first- and second-order boundary-layer equations governing the flow in the neighbourhood of a stagnation point of a three-dimensional body can be derived from the Navier–Stokes equations using the matched asymptotic expansion. Since the detailed derivation is presented in refs. [3, 4], here we write the equations in dimensionless form as:

*First-order equations*

$$f''' + (f+cg)f'' + 1 - f'^2 = 0 \quad (1a)$$

$$g''' + (f+cg)g'' + c - cg'^2 = 0 \quad (1b)$$

$$\theta'' + Pr(f+cg)\theta' = 0. \quad (1c)$$

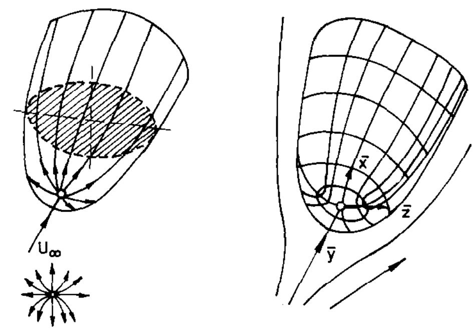


Fig. 1. Coordinate system.

The boundary conditions are

$$f(0) = f_w,\quad f'(0) = 0,\quad g(0) = 0,\quad g'(0) = 0,$$
$$\theta(0) = 0,\quad f'(\infty) = 1,\quad g'(\infty) = 1,\quad \theta(\infty) = 1 \quad (2)$$

where

$$\eta = U_{11}^{1/2} k_{x0} Re^{1/2} y,\quad \varepsilon = Re^{-1/2},\quad Re = U_\infty \rho / k_{x0}\mu,$$
$$c = W_{11}/U_{11} = [(\mathrm{d}W_1/\mathrm{d}z)/(\mathrm{d}U_1/\mathrm{d}x)]_0. \quad (3)$$

*Second-order equations*

1. Longitudinal curvature

$$D_1(F_L, G_L) = A_1 f'' + \eta(1-f'^2) + A_3 - c\chi + \eta(1-c) + cA_2 \quad (4a)$$

$$D_2(F_L, G_L) = g'(f+cg) + g'' + A_1 g'' - c\eta(1+g'^2) \quad (4b)$$

$$D_3(H_L) = -\theta' - Pr\,\theta'[F_L + cG_L - A_1]. \quad (4c)$$

Boundary conditions:

$$\eta = 0:\ F_L = F'_L = G_L = G'_L = H_L = 0 \quad (5a)$$

$$\eta \to \infty:\ F'_L \to -\eta,\ G'_L \to \eta,\ H_L \to 0. \quad (5b)$$

2. Transverse curvature

$$D_1(F_t, G_t) = A_1 f'' - \eta(1+f'^2) + f'(f+cg) + f'' \quad (6a)$$

$$D_2(F_t, G_t) = A_1 g'' + \eta c(1-g'^2) + A_3 - \chi + \eta(c-1) + A_2 \quad (6b)$$

$$D_3(H_t) = -\theta' - Pr\,\theta'[F_t + cG_t - A_1]. \quad (6c)$$

## NOMENCLATURE

| | |
|---|---|
| $c$ | ratio of velocity gradients |
| $C_{fx}, C_{fz}$ | skin friction coefficients in the $x$ and $z$ directions, respectively |
| $C_p$ | specific heat at a constant pressure |
| $f, g$ | dimensionless streamfunctions such that $f' = u/u_e$, $g = w/w_e$ |
| $f_w$ | mass transfer parameter, $-(\rho v)_w/(\varepsilon U_{11})^{1/2}$ |
| $f''(0), F''(0)$ | skin friction parameters in the $x$ direction |
| $g''(0), G''(0)$ | skin friction parameters in the $z$ direction |
| $H$ | dimensionless second-order temperature, $T/T_e$ |
| $H'(0)$ | second-order heat transfer parameter |
| $k$ | surface curvature of the body |
| $Pr, St, T$ | Prandtl number, Stanton number and temperature, respectively |
| $q_w$ | local heat transfer at the wall |
| $u, v, w$ | velocity components in the $x, y, z$ directions, respectively |
| $U_{11}, W_{11}$ | potential flow velocity gradients in the $x$ and $z$ directions, respectively |
| $x, y, z$ | principal, normal and transverse directions, respectively. |

Greek symbols

| | |
|---|---|
| $\varepsilon$ | the perturbation parameter, $Re^{-1/2}$ |
| $\eta$ | similarity variable |
| $\theta$ | dimensionless first-order temperature |
| $\theta'(0)$ | heat transfer parameter |
| $\mu, \rho$ | coefficient of viscosity and density, respectively |
| $\tau_x, \tau_z$ | shear stresses at the wall in the $x$ and $z$ directions, respectively |
| $\Omega_{x1}, \Omega_{z1}$ | vorticity interaction parameter in the $x$ and $z$ directions, respectively. |

Superscripts

| | |
|---|---|
| $'$ | derivative with respect to $\eta$. |

Subscripts

| | |
|---|---|
| d, D | the displacement effect terms proportional to $U_{21}$ and $W_{21}$, respectively |
| e, w | denote conditions at the edge of the boundary layer and on the surface, respectively |
| L, t | longitudinal and transverse curvature effect, respectively |
| v, V | vorticity interaction effect terms proportional to $\Omega_{z1}$ and $\Omega_{x1}$, respectively |
| $\infty$ | free-stream value. |

Boundary conditions:

$$\eta = 0: \ F_t = F_t' = G_t = G_t' = H_t = 0 \quad (7a)$$

$$\eta \to \infty: \ F_t' \to \eta, G_t' \to -\eta, H_t \to 0. \quad (7b)$$

3. Boundary-layer displacement (terms $\sim U_{21}$)

$$D_1(F_d, G_d) = -2; \quad D_2(F_d, G_d) = 0;$$

$$D_3(H_d) = A_4(F_d, G_d). \quad (8)$$

Boundary conditions

$$\eta = 0: \ F_d = F_d' = G_d = G_d' = H_d = 0 \quad (9a)$$

$$\eta \to \infty: \ F_d' \to 1, G_d' \to 0, H_d \to 0. \quad (9b)$$

The equations and boundary conditions for the boundary-layer displacement (terms $\sim W_{21}$) are same as equations (8) and (9) (by replacing $d$ by $D$) except that $D_1 = 0$; $D_2 = -2$ and

$$F_D' \to 0, G_D' \to 1/c, H_D \to 0 \quad \text{as } \eta \to \infty.$$

4. Vorticity interaction (terms $\sim \Omega_{z1}$)

$$D_1(F_v, G_v) = A_3; \quad D_2(F_v, G_v) = 0; \quad D_3(H_v) = A_4(F_v, G_v). \quad (10)$$

Boundary conditions

$$\eta = 0: \ F_v = F_v' = G_v = G_v' = H_v = 0 \quad (11a)$$

$$\eta \to \infty: \ F_v' \to -\eta, G_v' \to 0, H_v \to 0. \quad (11b)$$

The equations and the boundary conditions for the vorticity interaction (terms $\sim \Omega_{x1}$) are same as equations (10) and (11) (by replacing v by V) except that $D_1 = 0$, $D_2 = A_3$ and $F_V' \to 0$, $G_V' \to -\eta/c$, $H_V \to 0$ as $\eta \to \infty$. The operators $D_1$, $D_2$ and $D_3$ are

$$D_1(F, G) = F''' + (f+cg)F'' - 2f'F' + f''(F+cG) \quad (12a)$$

$$D_2(F, G) = G''' + (f+cg)G'' - 2cg'G' + g''(F+cG) \quad (12b)$$

$$D_3(H) = H'' + Pr(f+cg)H' \quad (12c)$$

and

$$\alpha = \lim_{\eta \to \infty} (\eta - f), \quad \beta = \lim_{\eta \to \infty} (\eta - g), \quad \chi = \int_0^\infty (1 - f'g')\, d\eta$$

$$A_1 = \eta(f+cg), \qquad A_2 = \int_0^\eta (1 - f'g')\, d\eta,$$

$$A_3 = \alpha + c\beta, \qquad A_4(F, G) = -Pr\,\theta'(F+cG). \quad (12d)$$

The components of the skin friction coefficients $C_{fx}$ and $C_{fz}$ and the heat transfer expressed by the Stanton number $St$ can be written in the following form [3, 4]:

$$C_{fx} = \tau_x/\rho U_\infty^2 = \varepsilon k_{x0} U_{11}^{3/2} x[f''(0) + \varepsilon U_{11}^{-1/2}(F_L''(0) + kF_t''(0)) + \varepsilon U_{11}^{-1}(U_{21}F_d''(0) + W_{21}F_D''(0)) + \varepsilon U_{11}^{-3/2}(\Omega_{z1}F_v''(0) + \Omega_{x1}F_V''(0))] \quad (13a)$$

$$C_{fz} = \tau_z/\rho U_\infty^2 = \varepsilon k_{x0} U_{11}^{3/2} cz[g''(0) + \varepsilon U_{11}^{-1/2}(G_L''(0) + lG_t''(0)) + \varepsilon U_{11}^{-1}(U_{21}G_d''(0) + W_{21}G_D''(0)) + \varepsilon U_{11}^{-3/2}(\Omega_{z1}G_v''(0) + \Omega_{x1}G_V''(0))] \quad (13b)$$

$$St = q_w/(\rho C_p U_\infty (T_\infty - T_w)) = \varepsilon Pr^{-1} U_{11}^{1/2}[\theta'(0) + \varepsilon U_{11}^{-1/2}(H_L'(0) + kH_t'(0)) + \varepsilon U_{11}^{-1}(U_{21}H_d'(0) + W_{21}H_D'(0)) + \varepsilon U_{11}^{-3/2}(\Omega_{z1}H_v'(0) + \Omega_{x1}H_V'(0))]. \quad (13c)$$

## RESULTS AND DISCUSSIONS

The set of first-order boundary-layer nonlinear equations (1) with boundary conditions (2) has been solved numerically using an implicit finite-difference scheme in combination with a quasilinearization technique in nodal point region. Since this method fails to work in the saddle-point region due to the occurrence of reverse flow in one of the velocity components, the method of parametric differentiation with the Runge–Kutta–Gill subroutine is used in the saddle-point region. The linear second-order boundary-layer equations with appropriate boundary conditions [i.e. equations (4)–(12)] have been solved using an implicit finite-difference scheme. Since these methods have been described in detail in refs. [6, 7], for the sake of brevity, they are not repeated here. Computations have been carried out for various values of the parameters. The step sizes $\Delta\eta$ and $\Delta c$ are optimized and

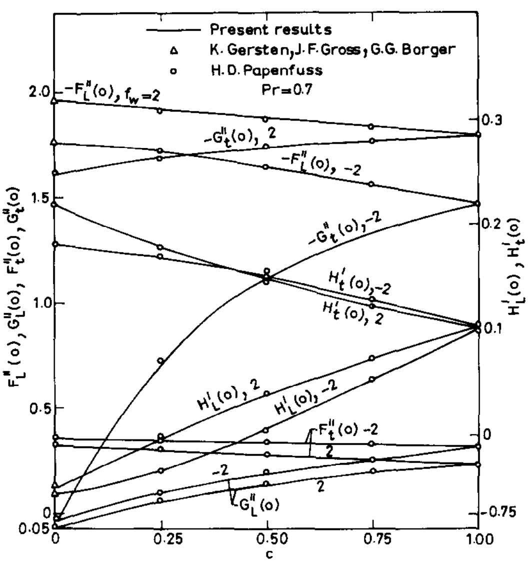


FIG. 2. Comparison of skin friction and heat transfer parameters (curvature effect).

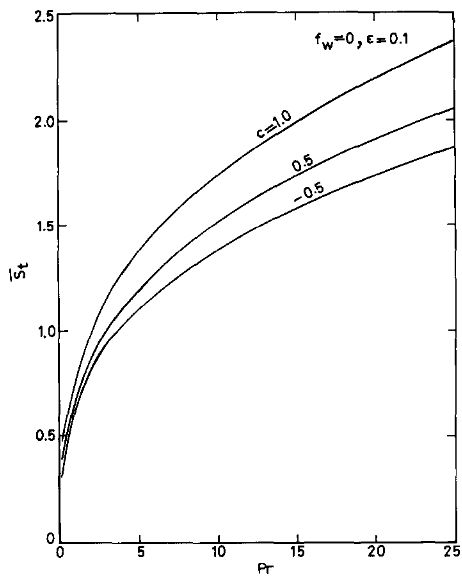


FIG. 4. Effect of $Pr$ on the heat transfer due to both first- and second-order boundary layers.

$\Delta\eta = 0.05$ and $\Delta c = -0.1$ are used throughout the computation.

In order to assess the accuracy of our method we have compared our first- and second-order boundary-layer results with those of Gersten *et al.* [5, 8] and Papenfuss [4, 9] and found them in good agreement. However, for the sake of brevity only comparisons with longitudinal and transverse curvature are shown (Fig. 2).

We assume that the $\Omega_{z1}$, $\Omega_{x1}$ and $U_{21}$ are negative and $0 \leqslant \Omega_{z1}, \Omega_{x1} \leqslant 0.6$ for $-1 \leqslant c \leqslant 1$ [10], $U_{21} \simeq -0.61$ [11, 12]. The value of $\varepsilon$ is taken to be 0.1.

The effect of mass transfer parameter $f_w$ on the skin friction $\bar{C}_{fx}$ $[\bar{C}_{fx} = (\varepsilon U_{11}^{3/2} k_{x0} x)^{-1} C_{fx}]$ and heat transfer $\overline{St}$ $(\overline{St} = (\varepsilon Pr^{-1} U_{11}^{1/2})^{-1} St)$ due to the combined effects of first- and second-order boundary layers [see equation (13)] is shown in Fig. 3. As expected, it is observed that both $\bar{C}_{fx}$ and $\overline{St}$ decrease due to injection ($f_w < 0$) and the effect of suction is just the opposite. The effect of $f_w$ on $\bar{C}_{fz}$ $[\bar{C}_{fz} = (\varepsilon U_{11}^{3/2} c k_{x0} z)^{-1} C_{fz}]$ is qualitatively similar to that on $\bar{C}_{fx}$, and hence is not shown here.

The effect of the Prandtl number $Pr$ on the heat transfer $\overline{St}$ due to the combined effects of both first- and second-order boundary layers is shown in Fig. 4. The heat transfer $\overline{St}$ is found to increase as the Prandtl number $Pr$ increases whatever the values of $c$ may be. The effect of $c$ on $\overline{St}$ becomes more pronounced as $Pr$ increases.

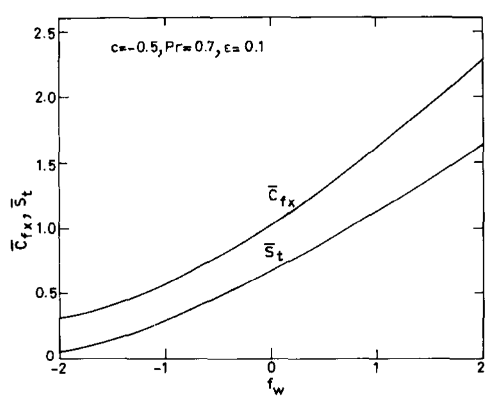


FIG. 3. Effect of $f_w$ on the skin friction and heat transfer due to both first- and second-order boundary layers.

## CONCLUSIONS

The effects of mass transfer (suction and injection) on the skin friction and heat transfer due to both first- and second-order boundary layers at the three-dimensional stagnation point have been studied. It is observed that both skin friction and heat transfer reduce as the injection rate increases, but the effect of suction is just the opposite. The heat transfer is found to increase as the Prandtl number increases.

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# Unsteady, three-dimensional, boundary-layer flow due to a stretching surface

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## 1. INTRODUCTION

THE FLOW and heat transfer problem due to a stretching boundary is important in extrusion processes. Tsou *et al.* [1] and Crane [2] among others have studied the steady flow problem caused by the two-dimensional stretching of a flat surface. Recently, a number of authors [3–7] have studied various aspects of this problem. More recently, Wang [8] considered the steady three-dimensional flow due to a stretching flat plate, where only the velocity field was studied.

The aim of the present analysis (which is an extension of Wang [8]) is to study the flow, heat and species transport problem due to the unsteady, three-dimensional flow caused by the stretching of a flat surface in two lateral directions. A self-similar solution has been obtained when the flat surface is stretched in a particular manner. The resulting nonlinear ordinary differential equations have been solved numerically [9].

## 2. GOVERNING EQUATIONS

We consider a highly elastic membrane immersed in a viscous fluid which is continuously stretched in the $x$ and $y$ directions and which also varies with time (see Fig. 1). The fluid velocities on the surface ($z = 0$) are given by:

$$u_w = ax(1-\lambda t^*)^{-1}, \quad v_w = by(1-\lambda t^*)^{-1}, \quad t^* = at. \quad (1)$$

The fluid has no lateral motions at $z \to \infty$. Also, it is assumed to have constant properties, and both wall and free stream are maintained at uniform temperature and concentration. The viscous dissipation term has been neglected. Here, we can confine our analysis to species diffusion processes in which the diffusion-thermal and thermo-diffusion effects can be neglected. The interfacial velocity at the wall $w_w$ due to mass diffusion process has also been neglected in the analysis. Under the foregoing assumptions, the unsteady boundary-layer equations governing the flow, and heat and diffusion transport can be expressed as:

$$u_x + v_y + w_z = 0 \quad (2)$$

$$u_t + uu_x + vu_y + wu_z = \nu u_{zz} \quad (3)$$

$$v_t + uv_x + vv_y + wv_z = \nu v_{zz} \quad (4)$$

$$T_t + uT_x + vT_y + wT_z = \alpha T_{zz} \quad (5)$$

$$C_t + uC_x + vC_y + wC_z = DC_{zz}. \quad (6)$$

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The initial and boundary conditions are given by

$$\left.\begin{aligned} &u(x,y,z,0) = u_i, \quad v(x,y,z,0) = v_i, \quad w(x,y,z,0) = w_i \\ &T(x,y,z,0) = T_i, \quad C(x,y,z,0) = C_i \end{aligned}\right\} \quad (7a)$$

$$\left.\begin{aligned} &u(x,y,0,t) = u_w, \quad v(x,y,0,t) = v_w, \quad w(x,y,0,t) = 0 \\ &T(x,y,0,t) = T_w, \quad C(x,y,0,t) = C_w \end{aligned}\right\} \quad (7b)$$

$$\left.\begin{aligned} &u(x,y,\infty,t) = v(x,y,\infty,t) = 0, \quad T(x,y,\infty,t) = T_\infty \\ &C(x,y,\infty,t) = C_\infty. \end{aligned}\right\} \quad (7c)$$

We apply the following transformations

$$\left.\begin{aligned} &\eta = (a/\nu)^{1/2}(1-\lambda t^*)^{-1/2}z, \quad \lambda t^* < 1, \quad c = b/a \\ &u = ax(1-\lambda t^*)^{-1}f'(\eta), \quad v = ay(1-\lambda t^*)^{-1}s'(\eta) \end{aligned}\right\} \quad (8a)$$

$$\left.\begin{aligned} &w = -(a\nu)^{1/2}(1-\lambda t^*)^{-1/2}(f+s), \quad Pr = \nu/\alpha, \quad Sc = \nu/D \\ &(T-T_\infty)/(T_w-T_\infty) = g(\eta), \quad (C-C_\infty)/(C_w-C_\infty) = G(\eta) \end{aligned}\right\} \quad (8b)$$

to equations (2)–(6) and we find that (2) is satisfied identically and equations (3)–(6) reduce to

$$f''' + (f+s)f'' - f'^2 - \lambda(f' + \eta f''/2) = 0 \quad (9)$$

$$s''' + (f+s)s'' - s'^2 - \lambda(s' + \eta s''/2) = 0 \quad (10)$$

$$Pr^{-1}g'' + (f+s)g' - \lambda\eta g'/2 = 0 \quad (11)$$

$$Sc^{-1}G'' + (f+s)G' - \lambda\eta G'/2 = 0. \quad (12)$$

The boundary conditions reduce to

$$\left.\begin{aligned} &f = s = 0, \quad f' = g = G = 1, \quad s' = c \quad \text{at } \eta = 0 \\ &f' = s' = g = G = 0 \quad \text{as } \eta \to \infty. \end{aligned}\right\} \quad (13)$$

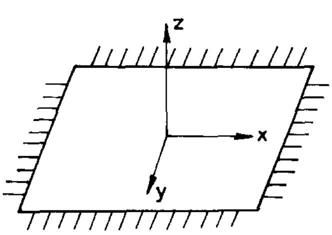


FIG. 1. Coordinate system.